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A Stable Limit Cycle Existence Analysis in a Diaphragm Thermo-Acoustic Oscillator Using Lyapunov Stability Theorem of Perturbed Systems

Azam Teymuri ^a, Aghileh Heydari ^a , Mohammad Reza Ramezani-al ^{b*}

^a Department of Mathematics, Payam Noor University, Tehran, Iran

^b Department of Electrical Engineering, Faculty of Electrical and Computer Engineering, Quchan University of Technology, Quchan, Iran

*Corresponding Author: m-ramezani@qiet.ac.ir

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Abstract

In this paper, the existence of a stable limit cycle in a diaphragm thermo-acoustic oscillator is analyzed using the Lyapunov stability theorem of perturbed dynamic systems. In this regard, first the dynamic differential equations of the thermo-acoustic oscillator are written as the state equations. Then, in order to system have a limit cycle, the error equations are extracted using the system equations and the desired states of the limit cycle. Next, a new Lyapunov function is introduced to asymptotic stability analysis of the error dynamics. Three conditions are examined, first condition of stability related to the positive definiteness of the Lyapunov function and the second condition related to the negativity of the derivative of this function. Third condition also guarantees that the error state is located within a certain bound to ensure that an asymptotic stable limit cycle occurs under these conditions. Besides, the upper and lower bounds of the error are obtained. Also, the effect of some important physical parameters of the system on the obtained error's bounds has been analyzed. The results obtained showed that the presented method has been able to successfully solve the most important challenge of this type of thermal oscillator, namely ensuring selecting the proper parameters for a thermo-acoustic oscillation.

Keywords

Diaphragm Thermo-acoustic Oscillator Nonlinear dynamics, Lyapunov stability, Limit cycle

1. Introduction

Most industrial efforts aim to stabilize the dynamics of engineering systems using various methods and theories [1]. However, some systems are intentionally designed to operate at the stability boundary, where the linearized model is marginally stable and highly sensitive to small disturbances [2]. Such systems are called oscillators and are widely used in medical devices, vehicle suspensions, laboratory equipment, musical instruments, simulators, bridges, and energy production systems [3]. Among these, energy production is particularly important: oscillations are transferred to generators to produce electrical energy [4]. Thermal oscillators convert heat into electrical energy and include Stirling, thermoacoustic, thermal phase-change (TPCO), and bimetallic types [5]. Thermo-acoustic oscillators are the newest class, categorized by the number of cores and the use of facilitators [6]. Their structure eliminates mechanical interfaces and the flywheel, offering high-frequency operation, high efficiency, and low maintenance, but also creating challenges in dynamic start-up [7]. Previous studies on thermal oscillators fall into two groups. Linear analyses have used dominant closed-loop poles, Nyquist diagrams, and genetic algorithms to study instability and threshold temperatures [8-10]. Nonlinear analyses have mainly relied on numerical solutions to study stable oscillations and parameter effects [11]. However, no analytical evaluation of the nonlinear dynamics of a diaphragm thermoacoustic oscillator has yet been reported. Therefore, this paper applies Lyapunov stability theory to analytically prove the existence of a stable limit cycle in a diaphragm thermo-acoustic oscillator.

2. Methodology

1.1 Dynamics of diaphragm audio thermal oscillator

Figure 1 illustrates the mechanical model of oscillator schematic in which, using Newton's second law, Equations (1) and (2) are derived as the force equations, and Equation (3) represents $F = \left(\frac{T_I}{T_O} - 1\right) D_1 (\dot{Z}_1 - \dot{Z}_2)$

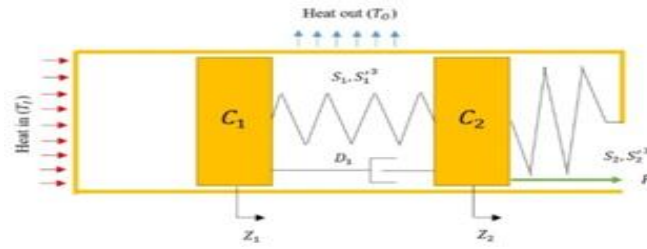


Figure 1. Mechanical model of a thermo-acoustic oscillator

$$C_1 \ddot{Z}_1 + D_1(\dot{Z}_1 - \dot{Z}_2) + S_1(Z_1 - Z_2) + S'_1(Z_1^3 - Z_2^3) = 0 \quad (1)$$

$$C_2 \ddot{Z}_2 - D_1(\dot{Z}_1 - \dot{Z}_2) - S_1(Z_1 - Z_2) - S'_1(Z_1^3 - Z_2^3) + S_2 Z_2 + S'_2 Z_2^3 = \left(\frac{T_l}{T_o} - 1\right) D_1(\dot{Z}_1 - \dot{Z}_2) \quad (2)$$

Finally, substituting F from Equation (2) into the dynamic equations and simplifying yields Equations (1) and (3).

$$C_2 \ddot{Z}_2 - \left(\frac{T_l}{T_o}\right) D_1 \dot{Z}_1 + \left(\frac{T_l}{T_o}\right) D_1 \dot{Z}_2 - S_1 Z_1 + (S_1 + S_2) Z_2 - S'_1 Z_1^3 + (S'_1 + S'_2) Z_2^3 = 0 \quad (3)$$

1.2 Analysis of the existence of the limit cycle and Lyapunov stability conditions for the Thermo-Acoustic Oscillator

Since the ultimate goal is to create a stable limit cycle in the oscillator dynamics, the optimal motion equations for the power piston and the displacement are introduced in Equations (4) and (5), which are as follows.

$$Z_{1d} = A_1 \sin(\omega t + \theta) \quad (4)$$

$$Z_{2d} = A_2 \sin(\omega t) \quad (5)$$

Actually, the objective is to bring the thermal oscillator's dynamics as close as possible to the intended motions given by Equations (4) and (5). Therefore, the error equations are defined as follows:

$$e_1 = Z_1 - A_1 \sin(\omega t + \theta) \quad (6)$$

$$e_2 = Z_2 - A_2 \sin(\omega t) \quad (7)$$

The dynamic equations governing the thermal oscillator (Equations (1) and (3)) are substituted into the error equations (6) and (7) and their derivatives to derive the dynamic error equation of the proposed thermos-acoustic oscillator

$$\begin{aligned} C_2 \ddot{e}_2 + \left(\frac{T_l}{T_o}\right) D_1 \dot{e}_2 - \left(\frac{T_l}{T_o}\right) D_1 \dot{e}_1 + (S_1 + S_2) e_2 - S_1 e_1 + (S'_1 + S'_2) e_2^3 + 3(S'_1 + \\ S'_2) e_2^2 A_2 \sin(\omega t) + 3(S'_1 + S'_2) e_2 (A_2 \sin(\omega t))^2 - S'_1 e_1^3 - 3S'_1 e_1^2 A_1 \sin(\omega t + \theta) - \\ 3S'_1 e_1 (A_1 \sin(\omega t + \theta))^2 = H_1(t) \end{aligned} \quad (8)$$

$$\begin{aligned} C_1 \ddot{e}_1 + D_1 \dot{e}_1 - D_1 \dot{e}_2 + S_1 e_1 - S_1 e_2 - S'_1 e_2^3 - 3S'_1 e_2^2 A_2 \sin(\omega t) - 3S'_1 e_2 (A_2 \sin(\omega t))^2 + \\ S'_1 e_1^3 + 3S'_1 e_1^2 A_1 \sin(\omega t + \theta) + 3S'_1 e_1 (A_1 \sin(\omega t + \theta))^2 = H_2(t) \end{aligned} \quad (9)$$

Which, will prove in the paper that $H_1(t)$ and $H_2(t)$ and boundness will hold.

The candidate Lyapunov function is given in the form of Equation (10).

$$V(e) = \frac{1}{2}S_1'e_1^4 + \frac{1}{2}S_2'e_2^4 + (S_1 + S_2)e_1^2 + S_1e_2^2 + C_2\dot{e}_1^2 + C_1\dot{e}_2^2 + \sqrt{(S_1 + S_2)C_2}e_1\dot{e}_1 - \sqrt{(S_1 + S_2)S_1}e_1e_2 - \sqrt{(S_1 + S_2)C_1}e_1\dot{e}_2 + \sqrt{S_1C_1}\dot{e}_2e_2 - \sqrt{C_2C_1}\dot{e}_2\dot{e}_1 \quad (10)$$

In the second step, based on the obtained differential equations, which are functions of the error vector and its derivatives, a candidate Lyapunov function is proposed. According to Lyapunov's theory, the positive definiteness of the Lyapunov function and the negative definiteness of its derivative are examined and analyzed. Subsequently, a suitable bound is found for the error state vector, which ensures that under this bound the system possesses an asymptotically stable limit cycle.

3. Results and Discussion

The simulation results were obtained using Simulink MATLAB 2023 and the ODE45 solver. We showed that the motion of the power piston (Figure 2) and the displacer piston (Figure 3) exhibit a stable limit cycle, and both pistons together possess a stable limit cycle (Figure 4). We have determined the bounds for the oscillator, such that the more this final bound can be reduced, the more stable the nonlinear dynamic oscillations become. In the continuation of this section, we will analyze the effect of some parameters, such as spring stiffness, cold space volume, hot space volume, cross-sectional area, and inlet and outlet temperatures, on the final bound value.

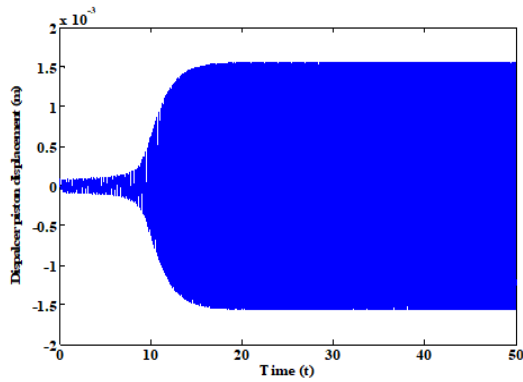


Figure 2. Power piston oscillation

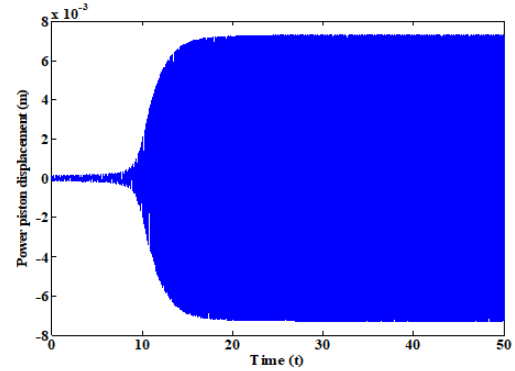


Figure 3. oscillatory movement of the displacement piston

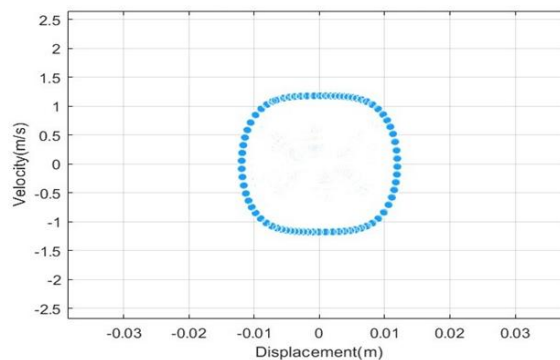


Figure 4. Stable limit cycle in the power and displacement piston

4. Conclusion

In this paper, using dynamic stability theory, a diaphragm thermoacoustic oscillator was analyzed, and the existence of a limit cycle in it was proved by establishing certain conditions. In fact, the purpose of this article is to provide a method for investigating the stable oscillations of this type of oscillator. In this regard, the diaphragm thermoacoustic oscillator was first introduced. The stability method was then applied to the nonlinear dynamics of this oscillator, and the proposed method was evaluated using the data of a built-in diaphragm thermal oscillator. Finally, the results were

analyzed. The outcome of this research is the provision of three parametric conditions; if these three conditions are satisfied, the stable oscillations of the studied thermal oscillator will be guaranteed, and consequently, the most important challenge faced by designers in this field will be resolved.

5. References

- [1] Adler, Robert. "A study of locking phenomena in oscillators." *Proceedings of the IRE* 34, no. 6 (1946): 351-357.
- [2] Jenkins, Alejandro. "Self-oscillation." *Physics Reports* 525, no. 2 (2013): 167-222.
- [3] Gammaitoni, Luca, Igor Neri, and Helios Vocca. "Nonlinear oscillators for vibration energy harvesting." *Applied Physics Letters* 94, no. 16 (2009).
- [4] Javad Faghihi, Mohammad, Mohammad Haddad, Zahra Mazaheri, and Hamid Reza Baghshahi. "Nonlinear thermal coherent states." *Journal of the Optical Society of America B* 37, no. 8 (2020): 2299-2307.
- [5]. Xuereb, André, Alberto Imparato, and Aurélien Dantan. "Heat transport in harmonic oscillator systems with thermal baths: application to optomechanical arrays." *New Journal of Physics* 17, no. 5 (2015): 055013.
- [6] Biwa, Tetsushi, Daichi Hasegawa, and Taichi Yazaki. "Low temperature differential thermoacoustic Stirling engine." *Applied Physics Letters* 97, no. 3 (2010).
- [7] Dashti, Ehsan, Nima Ghiyasi Tabari, Shahryar Zare, and Hanif Shabanpour. "An analytical investigation of a thermoacoustic stirling engine." *Arabian Journal for Science and Engineering* 49, no. 8 (2024): 11073-11090.
- [8] Bégot, Sylvie, Guillaume Layes, François Lanzetta, and Philippe Nika. "Stability analysis of free piston Stirling engines." *The European Physical Journal-Applied Physics* 61, no. 3 (2013): 30901..
- [9] Zare, S. H., and A. R. Tavakolpour-Saleh. "Frequency-based design of a free piston Stirling engine using genetic algorithm." *Energy* 109 (2016): 466-480.
- [10] Zare, Shahryar, and AliReza Tavakolpour-Saleh. "Design of a traveling wave thermo-acoustic engine based on genetic algorithm." *International Journal of Energy Research* 43, no. 14 (2019): 8790-8801.
- [11] Zare, Shahryar, and A. R. Tavakolpour-Saleh. "Modeling, construction, and testing of a diaphragm thermoacoustic Stirling engine." *Energy Conversion and Management* 243 (2021): 114394.